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Weighted Majoritarian Rules for the Location of Multiple Public Facilities

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Abstract

We consider collective decision problems given by a profile of single-peaked preferences defined over the real line and a set of pure public facilities to be located on the line. In this context, Bochet and Gordon (2012) provide a large class of priority rules based on *efficiency*, *object-population monotonicity* and *sovereignty*. Each such rule is described by a fixed priority ordering among interest groups. We show that any priority rule which treats agents symmetrically –*anonymity*–, respects some form of coherence across collective decision problems –*reinforcement*– and only depends on peak information –*peak-only*–, is a weighted majoritarian rule. Each such rule defines priorities based on the relative size of the interest groups and specific weights attached to locations. We give an explicit account of the richness of this class of rules.

Keywords: Multiple Public Facilities; Priority Rules; Weighted Majoritarian Rules; Object-Population Monotonicity; Sovereignty; Reinforcement; Anonymity.

JEL classification: D60; D63; D70; D71; H41.

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1 Introduction

We consider a generalization of the unidimensional voting model studied by Black (1948), Moulin (1980), and Barberà and Jackson (1994). A collective decision problem is given by a set of agents, a profile of single-peaked preferences defined over the real line, and a set of pure public facilities to be located on the line.[1] As is standard in the mechanism design literature, we look for rules which can solve any collective decision problem.

In this setup, Bochet and Gordon (2012) characterize a rich class of rules based on the combination of *efficiency*, *object-population monotonicity*, and *sovereignty*. While efficiency is a standard notion, the last two properties are new. Object-population monotonicity states that if newcomers join a collective decision problem and, at the same time, the number of public facilities increase to compensate for this arrival, then agents already in the initial problem cannot be hurt. Suppose next that a single facility must be located. Sovereignty states that any location could be chosen provided that an appropriately selected, and possibly large, interest group defending this particular location is brought into the problem. Each rule which jointly satisfies these three properties is a priority rule that selects locations based on a fixed priority ordering among interest groups.

An appealing feature of the class of priority rules is the simplicity with which these rules can be described. However, as will be made clear in Section 3, the class contains some rules which either give too much power to some agents, or exhibit inconsistencies across specific collective decision problems. We suggest to put some order in this class by imposing that a rule treat agents symmetrically – *anonymity* – and respect some form of coherence across collective decision problems – *reinforcement*.

Anonymity is a well-known property imposing that agents' label do not matter. Reinforcement is a property of stability with respect to merging of collective decision problems. It states that if for two problems – differing possibly on the cardinality of the set of agents and their preferences – the rule selects the same locations, then it should be invariant for the new collective decision problem obtained by merging the two initial problems. This property is, however, not new and already appears in the literature on characterizations of scoring rules – see e.g. Young (1975) or Myerson (1995). Along with a natural informational simplicity property

[1] By pure public facilities, we mean facilities which are non-excludable and do not suffer from congestion.

– *peak-only* –, any rule in Bochet and Gordon's class that satisfies anonymity and reinforcement is a weighted majoritarian rule. Each such rule defines priorities based on specific weights attached to interest groups. The weight of any interest group relative to another depends on their densities and most preferred locations. The simplest example is the rule which takes into account only the density of each interest group and gives priority to groups with the highest density. We call these simple majoritarian rules. However, the class is much larger. For instance, a rule can assign different weights to different interest groups based on the regions in which they are located. Also, rules can incorporate additional features such as the distance between the most preferred locations of the interest groups and a reference point. So while each rule in our class is "density-based", additional information can be used.

The problem of locating multiple public facilities was first introduced by Miyagawa (1998, 2001) in the case of two facilities. Ehlers (2002, 2003), Bochet and Gordon (2012), Heo (2012), Ehlers and Gordon (2011) provide axiomatic characterizations for this model. Umezawa (2012) considers the location of two facilities on a tree network. Barberà and Beviá (2002, 2005) and Ju (2008) show the existence of a rule satisfying interesting normative properties. Our main contribution to this literature is the analysis of the implications of the reinforcement axiom in this context and the characterization of the weighted majoritarian rules.

The plan of the paper is the following. In Section 2, we introduce the model. In Section 3, we introduce the properties we study, the class of priority rules and provide several illustrating examples. In Section 4, we prove our central result. Finally, we conclude in Section 5 by illustrating the richness of the characterized class of rules.

2 The Model and Notations

There is a countably infinite set $\mathbb{N}$ of potential agents. A *population* N is a finite and nonempty subset of $\mathbb{N}$. The population is collectively endowed with $k \geq 1$ identical public facilities, each to be located on the real line $\mathbb{R}$. A typical location on $\mathbb{R}$ is denoted by x. An *assignment* is a menu of locations, i.e., a finite subset $X \subset \mathbb{R}$. A k*-assignment* is an assignment for exactly k facilities, i.e., a subset $X \subset \mathbb{R}$ such that $|X| = k$. Let $\mathcal{X}_k$ be the class of all k-assignments. In particular, a 1-assignment is a single location $x \in \mathbb{R}$, so that $\mathcal{X}_1 = \mathbb{R}$. Let $\mathcal{X} \equiv \cup_{k \geq 1} \mathcal{X}_k$ be the class of all assignments.

Each agent $i \in N$ has a *preference* R_i over $\mathcal{X}$, which is a weak ordering (reflexive, transitive and complete) over $\mathcal{X}$. Let P_i and I_i be, respectively, the strict ordering and indifference relation derived from R_i. A preference R_i is *single-peaked* if the following hold:

i) There is a location $p(R_i)$, such that for all $x, y \in \mathbb{R}$ satisfying either $x < y \leq p(R_i)$ or $p(R_i) \geq y > x$, we have $y \; P_i \; x$. The location $p(R_i)$ is called the *peak* of preference R_i.

ii) For all $X, Y \in \mathcal{X}$, we let $X \; R_i \; Y$ if there is $x \in X$ such that for all $y \in Y$, we have $x \; R_i \; y$.

The first condition is the standard single-peakedness notion for preferences over single locations on the real line. The second condition extends the preferences from single locations to menus.[2] We restrict attention to the class $\mathcal{R}$ of single-peaked preferences over $\mathcal{X}$.

A *preference profile*, R_N, specifies a population N and the preferences of all agents in N, i.e., $R_N = (R_i)_{i \in N} \in \mathcal{R}^N$. For each profile R_N and each nonempty subpopulation $M \subseteq N$, let R_M denote the subprofile $(R_i)_{i \in M}$. For each profile R_N, let $p(R_N)$ be the set of *peak locations* for R_N, i.e., $p(R_N) \equiv \{p(R_i) : i \in N\}$. For each $k \geq 1$, let $\mathcal{P}_k$ be the set of preference profiles R_N such that $k \leq |p(R_N)|$, i.e., the number of distinct peak locations in R_N is at least k. A *problem* is a pair (k, R_N) such that $k \geq 1$ and $R_N \in \mathcal{P}_k$.[3]

A *rule* is a sequence $f = \{f_1, f_2, \ldots\}$ of mappings $f_k : \mathcal{P}_k \rightarrow \mathcal{X}_k$. For each problem (k, R_N), the rule f prescribes an assignment in $\mathcal{X}_k$.[4] For each $k \geq 1$, the set of mappings f_k is $\mathcal{X}_k^{\mathcal{P}_k}$. Therefore, the set of all rules is $\prod_{k=1}^{\infty} \mathcal{X}_k^{\mathcal{P}_k}$.

3 Main Axioms and Priority Rules

Consider a profile $R_N \in \mathcal{R}^N$ and $x, y \in \mathbb{R}$. For all $X, Y \in \mathcal{X}$, we say that X *weakly Pareto dominates* Y *for profile* R_N, denoted by $X \; R_N \; Y$, if $X \; R_i \; Y$ for each $i \in N$.

[2] There are different ways to extend preferences over points to preferences over sets. Consistent with the definition of a public facility used in this paper, we consider the max-extension of preferences used by Miyagawa (2001).

[3] The restriction $k \leq |p(R_N)|$ allows us to focus on non-trivial cases. When $k > |p(R_N)|$, it is possible to locate one facility at each peak location, so that the welfare of each agent is maximized. Locating the remaining facilities does not affect any agent's welfare.

[4] Our definitions rule out locating more than one facility at the same point. Under single-peaked preferences, and for the class of problems we consider, Pareto-efficiency would exclude duplication anyway.

Our first axiom is the usual *efficiency* axiom.

A rule f satisfies **efficiency** if, for each problem (k, R_N), there is no k-assignment X such that $X \; R_N \; f_k(R_N)$, and $X \; P_j \; f_k(R_N)$ for some $j \in N$.

A profile R_N is *peak-unanimous* if all preferences of this profile have the same peak, i.e., $p(R_N)$ is singleton. Let $\mathcal{T}$ be the set of peak-unanimous profiles.

A rule f satisfies **object-population monotonicity** if, for each problem (k, R_N) with $k < |p(R_N)|$, for each peak-unanimous profile $R_M \in \mathcal{T}$ such that $N \cap M = \emptyset$, we have $f_{k+1}(R_N, R_M) \; R_N \; f_k(R_N)$.

A rule f satisfies **sovereignty** if, for each profile R_N, each location $x \in \mathbb{R} \setminus f_1(R_N)$, and each population L, there exists a peak-unanimous profile $R_M \in \mathcal{T}$ such that M is disjoint from both L and N, and $f_1(R_N, R_M) = \{x\} = p(R_M)$.

On the one hand, in the situation of a population and resource increase, object-population monotonicity protects the rights of the first-comers. On the other hand, in the situation of a population increase, sovereignty protects the rights of the new-comers.

Bochet and Gordon (2012) show that the combination of efficiency, object-population monotonicity and sovereignty characterizes a subclass of priority rules. To define these rules, we need to introduce a class of binary relations called priorities over any nonempty subset $\mathcal{S}$ of $\mathcal{T}$. We say that any two peak-unanimous profiles R_N and R_M are *non-overlapping* if they have distinct peaks and disjoint populations, i.e., $p(R_N) \neq p(R_M)$ and $N \cap M = \emptyset$. The binary relation $\succ$ over $\mathcal{S}$ is *almost complete* if for all $R_N, R_M \in \mathcal{S}$, we have $(R_N \succ R_M$ or $R_M \succ R_N) \iff (R_N$ and R_M are non-overlapping).[5] It is *almost transitive* if for all profiles $R_N, R_M, R_L \in \mathcal{S}$, such that R_N and R_L are non-overlapping, we have $(R_N \succ R_M$ and $R_M \succ R_L) \implies (R_N \succ R_L)$. The binary relation $\succ$ is a *priority* over $\mathcal{S}$ if it is asymmetric, almost transitive and almost complete.[6] For each nonempty $\mathcal{S} \subseteq \mathcal{T}$, let $\mathbb{P}_{\mathcal{S}}$ be the set of priorities over $\mathcal{S}$.

For any profile R_N, the peak-unanimous subprofile R_M of R_N is *maximal* if $p(R_M) \cap p\left(R_{N \setminus M}\right) = \emptyset$. Since any two distinct maximal peak-unanimous subprofiles

[5] In particular, an almost complete binary relation $\succ$ over $\mathcal{T}$ is never reflexive.

[6] A priority $\succ$ is not a partial order, as it is not fully transitive. However, priorities have the following important property. The restriction of a priority $\succ$ on any set $\mathcal{S}$ of peak-unanimous and non-overlapping profiles is a strict ordering. If this set is finite, the priority $\succ$ has a greatest (or top) element in $\mathcal{S}$. A top element for $\succ$ typically does not generally exist on a set of peak-unanimous profiles whose elements are not non-overlapping, even if it is a finite set.

are non-overlapping, the set of maximal peak-unanimous subprofiles of some profile can be strictly ordered by any priority.

For each $\succ \in \mathbb{P}_{\mathcal{T}}$, the *priority rule f associated with $\succ$* is defined as follows. Let (k, R_N) be a problem. The priority $\succ$ strictly ranks the maximal peak-unanimous subprofiles in the decomposition of R_N and $f_k(R_N)$ selects the peak locations of the top k maximal peak-unanimous subprofiles for $\succ$. That is, $f_k(R_N)$ is the k-assignment such that $f_k(R_N) \subseteq p(R_N)$, and for all two maximal peak-unanimous subprofiles R_M and R_L in R_N, if $p(R_M) \subseteq f_k(R_N)$ and $p(R_L) \subsetneq f_k(R_N)$, then $R_M \succ R_L$. Let $\mathcal{F}$ be the class of priority rules.

We now introduce two properties that a priority ordering may satisfy. A priority $\succ$ is *almost monotonic* if there are no four peak-unanimous profiles R_M, R_K, R_H and R_L such that $p(R_M) = p(R_L)$, $M \cap L = \emptyset$, R_K and R_H are non-overlapping, $R_M \succ R_H \succ R_{M\cup L}$, and $R_M \succ R_K \succ R_{M\cup L}$. A priority $\succ$ is *sovereign* if the following two conditions hold. *(i)* For all peak-unanimous R_H, R_K such that $R_H \succ R_K$, and for any population L, there exists a peak-unanimous profile R_M such that M is disjoint from K and L, and satisfies $p(R_M) = p(R_K)$ and $R_{K\cup M} \succ R_H$. *(ii)* For each peak-unanimous profile R_H, each $x \neq p(R_H)$ and each population L, there exists a peak-unanimous profile R_M such that $M \cap L = \emptyset$, and satisfies $p(R_M) = x$ and $R_M \succ R_H$.

Next, we provide an example of a priority that is not sovereign.

Example 1 *Left-peaks priority / Right-peaks priority*
A priority $\succ$ is the left-peaks priority *if for all non-overlapping profiles $R_M, R_N \in \mathcal{T}$, we have $R_M \succ R_N \iff p(R_M) < p(R_N)$. Similarly, $\succ$ is the* right-peaks priority *if for all non-overlapping profiles $R_M, R_N \in \mathcal{T}$, we have $R_M \succ R_N \iff p(R_M) > p(R_N)$.* $\diamond$

We now state Bochet and Gordon (2012)'s central result.

Theorem 1 *A rule f satisfies efficiency, object-population monotonicity and sovereignty if and only if it is a priority rule whose priority is almost monotonic and sovereign.*

The proof of this result can be found in Bochet and Gordon (2012). We now give examples of priorities attached to rules described in Theorem 1.

Example 2 *Hierarchical priorities*

A priority $\succ$ is hierarchical if the following holds: (i) There is a weak ordering $\trianglerighteq$ of all agents in $\mathbb{N}$, such that, for all non-overlapping profiles R_L, $R_M \in \mathcal{T}$, if there exists $i \in L$, such that for all $j \in M$, we have ($i \trianglerighteq j$ and not $j \trianglerighteq i$), then $R_L \succ R_M$. (ii) For each $\trianglerighteq$-indifference class K, consider the class T_K of peak-unanimous profiles R_M such that the agents in M who are ranked highest according to $\trianglerighteq$ belong to K. On each such class T_K, the priority coincides with either the left-peaks or the right-peaks priority. If each $\trianglerighteq$-indifference class is a singleton, the priority is a serial dictatorship. Also, if there is a single $\trianglerighteq$-indifference class, the priority is either the left-peaks or the right-peaks priority. $\diamond$

Note that a hierarchical priority, as described in Example 2, is sovereign (and therefore satisfies all the properties in Theorem 1) if and only if the weak ordering $\trianglerighteq$ has no maximal element.

Example 3 *Simple majoritarian priorities*
A priority $\succ$ is simple majoritarian if for all non-overlapping R_L, $R_M \in \mathcal{T}$, we have $|L| > |M| \Rightarrow R_L \succ R_M$. For each $n \geq 1$, the tie-breaking rule $\succ_n$ within each class of the form $T_n = \{R_N \in \mathcal{T} : |N| = n\}$ can be given by any strict ordering over locations in $\mathbb{R}$. For example, we could require $\succ_n$ to be the left-peaks priority for all n (left majoritarian priority) or the right-peaks priority for all n (right majoritarian priority). $\diamond$

Unlike the rules described by simple majoritarian priorities, the rules described by hierarchical priorities allow for an asymmetric treatment of agents, i.e., agents' labels matter. We would like rules to respect an anonymous treatment of agents' preferences.

A rule f satisfies **anonymity** if for all $k \geq 1$ and problems (k, R_N) and (k, R'_M) such that for all $R \in \mathcal{R}$, $|\{i \in M : R'_i = R\}| = |\{i \in N : R_i = R\}|$, we have $f_k(R_N) = f_k(R'_M)$.

Anonymity imposes an additional requirement on priorities. A priority $\succ$ is *anonymous* if it satisfies the following condition. For all $R_M, R_N, R'_{M'}, R'_{N'} \in \mathcal{T}$, such that (i) R_M and R_N are non-overlapping, (ii) $R'_{M'}$ and $R'_{N'}$ are non-overlapping, (iii) for all $R \in \mathcal{R}$, we have $|\{i \in M : R_i = R\}| = |\{i \in M' : R'_i = R\}|$, (iv) for all $R \in \mathcal{R}$, we have $|\{i \in N : R_i = R\}| = |\{i \in N' : R'_i = R\}|$, the following equivalence holds, $R_M \succ R_N \iff R'_{M'} \succ R'_{N'}$.

Bochet and Gordon (2012) characterized the subclass of anonymous priority rules. We state it below and omit its straightforward proof.

Theorem 2 *Let f be a priority rule. Then f satisfies anonymity if and only if its priority is anonymous.*

Notice that if anonymity is dropped, the class of rules which satisfy efficiency, object-population monotonicity and sovereignty will include rules whose priorities combine Examples 2 and 3 in interesting ways. We give below two such examples.

Example 4 *Majoritarian-hierarchical priorities*
Let $\trianglerighteq$ be a weak ordering over agents in $\mathbb{N}$, such that for all i, j, $i < j \implies j \trianglerighteq i$ and, in addition, $\trianglerighteq$ has no maximal element. Construct the partition of $\mathbb{N}$ into indifference classes $Z_1, Z_2, ...$ according to $\trianglerighteq$. That is, for each $i, j \in Z_k$, $i \trianglerighteq j$ and $j \trianglerighteq i$. In addition, for each $i \in Z_k$, $j \in Z_\ell$ with $k < \ell$, we have $j \trianglerighteq i$.
A priority $\succ$ is a majoritarian-hierarchical priority if there is an index u defined on the class of all populations, such that for each population $M \subset \mathbb{N}$, u is defined as,

$$u(M) = \delta|M| + (1-\delta)\max\{k \in \mathbb{N} : Z_k \cap M \neq \emptyset\}$$

for $\delta \in (0,1)$, and for all non-overlapping $R_M, R_N \in \mathcal{T}$, we have that $u(M) > u(N) \implies R_M \succ R_N$. For each $v > 0$, the tie-breaking rule $\succ_v$ within each class of the form $T_v = \{R_N \in \mathcal{T} : u(N) = v\}$ is either the left-peaks or the right-peaks priority. ◇

Example 5 *Hierarchical weighted majoritarian priorities*
A priority $\succ$ is a hierarchical weighted majoritarian priority if there exists a list of weights $(\omega_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with

$$\sum_{i=1}^{\infty} \omega_i = +\infty,$$

such that for all non-overlapping $R_M, R_N \in \mathcal{T}$, we have

$$\sum_{i\in M} \omega_i > \sum_{i \in N} \omega_i \implies R_M \succ R_N.$$

For cases where equality holds, the tie-breaking rule $\succ$ within each v level curve of the form $\{R_N \in \mathcal{T} : \sum_{i\in N} \omega_i = v\}$ is determined by some strict ordering $\triangleright$ over locations, independent of v.[7] ◇

In Bochet and Gordon (2012), it is shown that the set of priority rules described by hierarchical priorities is equivalent to the set of strategy-proof priority rules.

[7] Note that a tie-breaking rule may not be needed if equality never holds for any two non-overlapping peak-unanimous profiles.

In contrast, there are only two hierarchical priorities that are anonymous: left-peaks and right-peaks priorities. But left-peaks and right-peaks priorities are not sovereign. Thus, anonymity, sovereignty and strategy proofness are mutually inconsistent within the class of priority rules. If we do not impose anonymity, then the class of rules characterized by all other properties (i.e. efficiency, object-population monotonicity, sovereignty, reinforcement and peak-only) will include every hierarchical rule whose associated priority is sovereign.[8] Adding anonymity excludes rules like serial dictatorship that violate the pure notion of majoritarianism. But it also excludes some rules which balance agents' priorities with the notion of majoritarianism –e.g. the priority introduced in Example 5.[9]

We now introduce additional examples of priority rules which also satisfy anonymity.

Example 6 *Two-regions majoritarian priorities*
A priority $\succ$ *is two-regions majoritarian if there is a location* $x_0 \in \mathbb{R}$ *(that separates the two regions), and a coefficient* $\lambda_{x_0} \in (0,1]$ *such that, for all non-overlapping* R_M*,* $R_N \in \mathcal{T}$*, if either (i)* $p(R_M) < x_0 \leq p(R_N)$ *and* $\lambda_{x_0} |M| > |N|$ *or (ii)* $p(R_N) < x_0 \leq p(R_M)$ *and* $|M| > \lambda_{x_0} |N|$ *or (iii)* $p(R_M), p(R_N) \in (-\infty, x_0)$*, or* $p(R_M), p(R_N) \in [x_0, +\infty)$ *and* $|M| > |N|$ *hold, then* $R_M \succ R_N$*. For each* $v \geq \lambda_{x_0}$*, the tie-breaking rule* $\succ_v$ *within each class of the form*

$$T_v = \{R_N \in \mathcal{T} : p(R_N) \geq x_0,\ |N| = v\} \cup \{R_N \in \mathcal{T} : p(R_N) < x_0,\ \lambda_{x_0} |N| = v\}$$

can be given by any strict ordering over locations in $\mathbb{R}$*. For example, we could require* $\succ_v$ *to be the left-peaks priority for all* T_v *(left-two-regions majoritarian priority) or the right-peaks priority for all* T_v *(right-two-regions majoritarian priority).* ◇

Example 7 *Centralist majoritarian priorities*
A priority $\succ$ *is centralist majoritarian if there is a location* $x_0 \in \mathbb{R}$ *(the "center") and an index* $u : \{1, 2, ...\} \times \mathbb{R}_+ \to \mathbb{R}$*, where* $u(n, d)$ *is weakly increasing in* n *and weakly decreasing in* d*, with* $\lim_{n \to +\infty} u(n, d) = +\infty$*, such that for all non-overlapping* R_M*,* $R_N \in \mathcal{T}$*, we have*

$$u(|M|, |p(R_M) - x_0|) > u(|N|, |p(R_N) - x_0|) \Rightarrow R_M \succ R_N.$$

[8] Suppose we do not impose anonymity but add strategy-proofness instead to efficiency, object-population monotonicity and sovereignty. The class of rules characterized by these four axioms coincides with hierarchical rules whose priorities are sovereign. Notice that hierarchical priority rule whose priorities are sovereign satisfy reinforcement and peak-only. Adding reinforcement and peak-only has thus no refining effect on this characterization.

[9] The priority in Example 4 does not satisfy both anonymity and reinforcement.

For each $v \in \mathbb{R}$, the tie-breaking rule $\succ_v$ within each class of the form

$$T_v = \{R_N \in \mathcal{T} : u(|N|, |p(R_N) - x_0|) = v\}$$

can be any strict ordering over locations in $\mathbb{R}$. For example, we could require $\succ_v$ to be the left-peaks priority for all T_v (left-centralist majoritarian priority) or the right-peaks priority for all T_v (right-centralist majoritarian priority). $\diamond$

There are many possible functions u for a centralist majoritarian priority rule. For example, with

$$u(n,d) = \begin{cases} \frac{n}{\delta+d} & \text{if } n \leq 2 \\ \max\left\{n, \frac{2}{\delta+d}\right\} & \text{if } n > 2, \end{cases}$$

where $\delta > 0$, the priority rule f behaves across problems in a way that is not coherent. That is, if for two problems – differing possibly on the cardinality of the set of agents, and on preferences – the rule selects the same locations, then the selection operated by f may change for the new collective decision problem obtained by merging the two initial problems. For instance, let $x_0 = 1$, $\delta = 0.1$ and consider the problems $(1, R_M)$ and $(1, R_L)$ with $M \cap L = \emptyset$, $M = H \cup K$, $|H| = 1$, $|K| = 2$, $p(R_H) = \frac{1}{2}$, $p(R_K) = 3$; $L = H' \cup K'$, $|H'| = 2$, $|K'| = 3$, $p(R_{H'}) = \frac{1}{2}$, $p(R_{K'}) = 3$. It is easy to see that, given u, $f_1(R_M) = f_1(R_L) = \{\frac{1}{2}\}$. However, $f_1(R_M \cup R_L) = \{3\}$.

We are interested in rules with the following coherence property: if two problems deliver the same location(s), the problem obtained from merging them still delivers the same location(s).

A rule f satisfies **reinforcement** if for all $k \geq 1$ and each pair of profiles R_N, R_M such that $N \cap M = \emptyset$, if $f_k(R_N) = f_k(R_M) = X$ then $f_k(R_N \cup R_M) = X$.

That is, whenever any two different problems (k, R_N) and (k, R_M) select the same k locations, then reinforcement requires that the location of the k facilities should not change in the problem $(k, R_N \cup R_M)$.

We add one last property that will be used for our main result.

A rule f satisfies **peak-only** if for all $k \geq 1$ and problems (k, R_N) and (k, R'_N), if $p(R_i) = p(R'_i)$ for all $i \in N$, then $f_k(R_N) = f_k(R'_N)$.

Peak-only is an informational simplicity requirement which states that only the information regarding the peaks of agents should be used. It is, however, a strong assumption as it ignores every other aspect of agents' preference orderings. Nevertheless, if we do not impose peak-only, then the class of priority rules characterized by all other properties (i.e. efficiency, object-population monotonicity, sovereignty, anonymity and reinforcement) will include rules that put arbitrary weights on preference orderings. This forms a rich class of rules where weights can depend in complicated ways on the full preference relations. Examples of such rules are those described by the following priorities:

Example 8 *Symmetry biased majoritarian priorities*
We say that a single-peaked preference R_i is symmetric if for all $x, y \in \mathbb{R}$, we have $xR_iy \iff |x - p(R_i)| \leq |y - p(R_i)|$. For any peak-unanimous profile $R_N \in \mathcal{T}$, let $\gamma(R_N)$ be the number of agents $i \in N$ such that R_i is symmetric.

A priority $\succ$ is symmetry biased majoritarian if there is $\delta > 0$ such that for all non-overlapping R_L, $R_M \in \mathcal{T}$, we have $\delta\gamma(R_L) + |L| > \delta\gamma(R_M) + |M| \Rightarrow R_L \succ R_M$. For each $n \geq 1$, the tie-breaking rule $\succ_n$ within each class of the form $T_n = \{R_N \in \mathcal{T} : \delta\gamma(R_N) + |N| = n\}$ can be given by any strict ordering over locations in $\mathbb{R}$. For example, we could require $\succ_n$ to be the left-peaks priority for all n or the right-peaks priority for all n. $\diamond$

Thus, imposing peak-only excludes "undesirable" rules like those described by symmetry biased majoritarian priorities. Our position here is that peak-only is a relevant requirement when agents' peaks (but not preferences) are commonly known. Indeed, in many instances, peak information is difficult to manipulate because it reflects some observable attributes –e.g. because it reflects an agent's address. Under this interpretation, peak-only is an invariance condition with respect to some preference change.[10]

4 A Characterization of Weighted Majoritarian Rules

We now introduce a family of priority rules that we call weighted majoritarian rules. Let $\mathbb{R}_{++}$ and $\mathbb{Q}_{++}$ respectively be the set of positive reals and the set of positive rationals.

[10] But then, notice that the requirement of strategy-proofness in this model may not be appropriate. Indeed, if peaks are verifiable, a weakening of strategy-proofness is called for. Sakai and Wakayama (2012) introduce such a weakening, *strategy-proofness for same peaks*, which preclude manipulations of preference relations around the true peak. Notice that strategy-proofness for same peaks is implied by peak-only.

A priority $\succ$ is a **weighted majoritarian priority** if there exists an asymmetric and transitive binary relation, i.e. a strict partial order $\rhd$ on $\mathbb{R}$, and a function $q : \mathbb{R}^2 \to \mathbb{R}_{++}$ with $q(x,y)q(y,x) = 1$, $q(x,z) = q(x,y)q(y,z)$, and $q(x,y) \in \mathbb{Q}_{++} \iff$ (either $x \rhd y$ or $y \rhd x$), for all distinct x, y and z, such that for any two non-overlapping peak-unanimous profiles R_M and R_L, we have $R_M \succ R_L$ if either

1. $\frac{|M|}{|L|} > q(p(R_M), p(R_L))$, or
2. $\frac{|M|}{|L|} = q(p(R_M), p(R_L))$ and $p(R_M) \rhd p(R_L)$.

Note that the tie-breaking rule $\rhd$ is only needed if the image of q contains at least one rational number; otherwise, the equality $\frac{|M|}{|L|} = q(p(R_M), p(R_L))$ does not hold for any two peak-unanimous profiles R_M and R_L.

Theorem 3 *A rule f satisfies efficiency, object-population monotonicity, sovereignty, anonymity, reinforcement and peak-only if and only if there exists a weighted majoritarian priority $\succ$ such that f is priority rule associated with $\succ$.*

Proof. It is straightforward to prove the if part, i.e., if there exists a weighted majoritarian priority $\succ$ such that f is a priority rule associated with $\succ$, then f satisfies all the axioms listed in the theorem. We prove the only if part.

It follows from Theorem 1 that if f satisfies efficiency, object-population monotonicity and sovereignty, then there exists a priority $\succ$ such that f is a priority rule associated with $\succ$. We show that $\succ$ is a weighted majoritarian priority.

Pick any two locations $x, y \in \mathbb{R}$ such that $x > y$.

Step 1. Let R_M, R'_N, R_K, R'_L be four peak-unanimous preference profiles such that $p(R_M) = p(R'_N) = x$, $p(R_K) = p(R'_L) = y$, $|M| = |N|$, $|L| = |K|$, and both pairs (R_M, R_K) and (R'_N, R'_L) are non-overlapping. By efficiency $f_1(R_M) = f_1(R'_N) = \{x\}$ and $f_1(R_K) = f_1(R'_L) = \{y\}$. By peak-only and anonymity, we get that $f_1(R_M, R_K) = f_1(R'_N, R'_L) \subset \{x, y\}$, where the set inclusion follows because f is a priority rule.

Pick a $(n_1, n_2) \in \mathbb{Z}^2_+ \backslash (0,0)$, where $\mathbb{Z}_+$ is the set of nonnegative integers. If $n_1 > 0$ and $n_2 = 0$, then let R_{N_1} be any peak-unanimous profile such that $p(R_{N_1}) = x$ and $|N_1| = n_1$. If $n_1 = 0$ and $n_2 > 0$, then let R_{N_2} be any peak-unanimous profile such that $p(R_{N_2}) = y$ and $|N_2| = n_2$. If $n_1, n_2 > 0$, then let (R'_{N_1}, R'_{N_2}) be any pair of peak-unanimous and non-overlapping profiles such that $p(R'_{N_1}) = x$, $p(R'_{N_2}) = y$,

$|N_1| = n_1$ and $|N_2| = n_2$. Define

$$g_{xy}(n_1, n_2) = \begin{cases} f_1(R_{N_1}), & \text{if } n_2 = 0 \\ f_1(R_{N_2}), & \text{if } n_1 = 0 \\ f_1(R'_{N_1}, R'_{N_2}), & \text{if } n_1, n_2 > 0. \end{cases}$$

The argument in the previous paragraph implies that g_{xy} is a well-defined function over the domain $\mathbb{Z}_+^2 \backslash (0,0)$.

Now, we extend the domain of $g_{(x,y)}$ from $\mathbb{Z}_+^2 \backslash (0,0)$ to $\mathbb{Q}_+^2 \backslash (0,0)$, where $\mathbb{Q}_+$ is the set of nonnegative rational numbers. For any positive integer n define $g_{xy}\left(\frac{n_1}{n}, \frac{n_2}{n}\right) = g_{xy}(n_1, n_2)$. This is well-defined because for any two $\left(\frac{n_1}{n}, \frac{n_2}{n}\right) = \left(\frac{n'_1}{n'}, \frac{n'_2}{n'}\right)$, we have

$$g_{xy}\left(\frac{n_1}{n}, \frac{n_2}{n}\right) = g_{xy}(n_1, n_2) = g_{xy}\left(\frac{n \times n'_1}{n'}, \frac{n \times n'_2}{n'}\right) = g_{xy}(n \times n'_1, n \times n'_2) = g_{xy}(n'_1, n'_2),$$

where the last equality follows from reinforcement. Note that by this extension, g_{xy} is defined for any pair of rational numbers $(q_1, q_2) \in \mathbb{Q}_+^2 \backslash (0,0)$ since any such (q_1, q_2) equals $\left(\frac{z_1}{z}, \frac{z_2}{z}\right)$, where z_1, z_2 are nonnegative integers while z is a positive integer.

Pick any $\left(\frac{n_1}{n}, \frac{n_2}{n}\right), \left(\frac{n'_1}{n'}, \frac{n'_2}{n'}\right) \in \mathbb{Q}_+^2 \backslash (0,0)$ such that $\frac{n_1}{n} < \frac{n'_1}{n'}$ and $\frac{n_2}{n} = \frac{n'_2}{n'}$. We argue that if $g_{xy}\left(\frac{n_1}{n}, \frac{n_2}{n}\right) = \{x\}$, then $g_{xy}\left(\frac{n'_1}{n'}, \frac{n'_2}{n'}\right) = \{x\}$. To prove this, let $\frac{\tilde{n}_1}{\tilde{n}} = \frac{n'_1}{n'} - \frac{n_1}{n}$. Now,

$$\begin{aligned} g_{xy}\left(\frac{n'_1}{n'}, \frac{n'_2}{n'}\right) = g_{xy}(n'_1, n'_2) &= g_{xy}\left(n \times \tilde{n} \times n'_1, n \times \tilde{n} \times n'_2\right) \\ &= g_{xy}\left(n'(\tilde{n} \times n_1 + \tilde{n}_1 \times n), n' \times \tilde{n} \times n_2\right) \\ &= g_{xy}\left(\tilde{n} \times n_1 + \tilde{n}_1 \times n, \tilde{n} \times n_2\right) \\ &= g_{xy}\left((\tilde{n} \times n_1, \tilde{n} \times n_2) + (\tilde{n}_1 \times n, 0)\right), \end{aligned}$$

where the second and the fourth equalities follow from reinforcement. However, $g_{xy}(\tilde{n} \times n_1, \tilde{n} \times n_2) = g_{xy}(n_1, n_2) = g_{xy}\left(\frac{n_1}{n}, \frac{n_2}{n}\right) = \{x\}$ (the first equality follows from reinforcement) and $g_{xy}(\tilde{n}_1 \times n, 0) = \{x\}$. Once again, reinforcement implies that $g_{xy}\left((\tilde{n} \times n_1, \tilde{n} \times n_2) + (\tilde{n}_1 \times n, 0)\right) = \{x\}$ and so we are done.

Step 2. Define

$$q^+(x,y) = \sup\{q_1 \in \mathbb{Q}_+ : g_{xy}(q_1, 1) = \{y\}\}$$
$$q^-(x,y) = \inf\{q_1 \in \mathbb{Q}_+ : g_{xy}(q_1, 1) = \{x\}\}.$$

We argue that $\infty > q^+(x,y) = q^-(x,y) > 0$. It is easy to see that $q^+(x,y) < \infty$ since sovereignty implies that there exists an integer $n_1 > 0$ such that $g_{xy}(n_1, 1) = \{x\}$ and the last result in Step 1 implies that $g_{xy}(q_1, 1) = \{x\}$ for all rational $q_1 \geq n_1$. Likewise, $q^-(x,y) > 0$ since sovereignty implies that there exists an integer $n_2 > 0$ such that $g_{xy}(1, n_2) = \{y\}$. However, $g_{xy}(1, n_2) = g_{xy}\left(\frac{1}{n_2}, 1\right)$ and so the last result in Step 1 implies that $g_{xy}(q_1, 1) = \{y\}$ for all rational $q_1 \leq \frac{1}{n_2}$.

It must be that $q^+(x,y) \leq q^-(x,y)$ because otherwise there exists a $q_1 \in \mathbb{Q}_+$ such that $q^-(x,y) < q_1 < q^+(x,y)$. If $g_{xy}(q_1, 1) = \{x\}$, then the last result in Step 1 implies that $g_{xy}(q_1', 1) = \{x\}$ for all $q_1' > q_1$ and therefore, we must have $q^+(x,y) \leq q_1$, a contradiction. Similarly, if $g_{xy}(q_1, 1) = \{y\}$, then $g_{xy}(q_1', 1) = \{y\}$ for all $q_1' < q_1$ and therefore, we must have $q_1 \leq q^-(x,y)$, a contradiction. Now, suppose $q^+(x,y) < q^-(x,y)$ and let $q_1 \in \mathbb{Q}_+$ such that $q^+(x,y) < q_1 < q^-(x,y)$. By definition of $q^+(x,y)$, it must be that $g_{xy}(q_1, 1) = \{x\}$ whereas by definition of $q^-(x,y)$ it must be that $g_{xy}(q_1, 1) = \{y\}$, a contradiction. Hence, we conclude that $q^+(x,y) = q^-(x,y)$.

Define $q(x,y) = q^+(x,y) = q^-(x,y)$ and $q(y,x) = \frac{1}{q(x,y)}$. Next, define the binary relation $\triangleright$ as follows: if $q(x,y)$ is irrational, then x and y are not comparable for $\triangleright$. If $q(x,y)$ is rational and $g_{xy}(q(x,y), 1) = \{x\}$, then $x \triangleright y$, whereas if $q(x,y)$ is rational and $g_{xy}(q(x,y), 1) = \{y\}$, then $y \triangleright x$.

Step 3. Pick any two peak-unanimous and non-overlapping profiles R_M and R_L such that $p(R_M) = x$ and $p(R_L) = y$. Since f is a priority rule associated with $\succ$, we know that $f_1(R_M, R_L) = \{x\} \iff R_M \succ R_L$.

In Step 1, we have argued that $f_1(R_M, R_L) = g_{xy}(|M|, |L|) = g_{xy}\left(\frac{|M|}{|L|}, 1\right)$.

By definition of the function $q(.,.)$, it follows that if $\frac{|M|}{|L|} > q(x,y)$ (or equivalently $\frac{|L|}{|M|} < q(y,x)$), then $g_{xy}\left(\frac{|M|}{|L|}, 1\right) = \{x\}$ and therefore, $R_M \succ R_L$. Similarly, if $\frac{|M|}{|L|} < q(x,y)$ (or equivalently $\frac{|L|}{|M|} > q(y,x)$), then $g_{xy}\left(\frac{|M|}{|L|}, 1\right) = \{y\}$ and therefore, $R_L \succ R_M$. Finally, if $\frac{|M|}{|L|} = q(x,y)$ (or equivalently $\frac{|L|}{|M|} = q(y,x)$), then $x \triangleright y \iff g_{xy}(q(x,y), 1) = \{x\} \iff R_M \succ R_L$.

Step 4. Next, we argue that $q(x,z) = q(x,y)q(y,z), \forall x \neq y \neq z$. Let $\left(\frac{n_1(n)}{\tilde{n}_1(n)}\right)_{n=1}^{\infty}$ be a sequence of rational numbers such that $\frac{n_1(n)}{\tilde{n}_1(n)} \geq q(x,y)$ and $\lim_{n\to\infty} \frac{n_1(n)}{\tilde{n}_1(n)} = q(x,y)$. Similarly, let $\left(\frac{n_2(n)}{\tilde{n}_2(n)}\right)_{n=1}^{\infty}$ a sequence of rational numbers such that $\frac{n_2(n)}{\tilde{n}_2(n)} \geq q(y,z)$ and $\lim_{n\to\infty} \frac{n_2(n)}{\tilde{n}_2(n)} = q(y,z)$. Let R_{M_n}, R_{L_n} and R_{K_n} be three peak-unanimous and non-overlapping profiles such that $p(R_{M_n}) = x$, $p(R_{L_n}) = y$ and $p(R_{K_n}) = z$, and $|M_n| = n \times n_1(n) \times n_2(n) + 2 \times n_1(n) \times n_2(n)$, $|L_n| = n \times n_2(n) \times \tilde{n}_1(n) + n_2(n)$

and $|K_n| = n \times \tilde{n}_1(n) \times \tilde{n}_2(n)$. Consider the problem $(1, (R_{M_n}, R_{L_n}, R_{K_n}))$. We have $\frac{|M_n|}{|L_n|} = \frac{n \times n_1(n) \times n_2(n) + 2 \times n_1(n) \times n_2(n)}{n \times n_2(n) \times \tilde{n}_1(n) + n_2(n)} > \frac{n_1(n)}{\tilde{n}_1(n)} \geq q(x,y)$ and $\frac{|L_n|}{|K_n|} = \frac{n_2(n)}{\tilde{n}_2(n)} + \frac{n_2(n)}{n \times \tilde{n}_1(n) \times \tilde{n}_2(n)} > q(y,z)$. Therefore, from the arguments in Step 3, it follows that $R_{M_n} \succ R_{L_n}$ and $R_{L_n} \succ R_{K_n}$. Then we must have $R_{M_n} \succ R_{K_n}$ since $\succ$ is a priority, which is almost transitive. This implies that $\frac{|M_n|}{|K_n|} \geq q(x,z), \forall n$. However, $\lim_{n \to \infty} \frac{|M_n|}{|K_n|} = q(x,y)q(y,z)$, and therefore, $q(x,y)q(y,z) \geq q(x,z)$. We can similarly argue that $q(z,y)q(y,x) \geq q(z,x) \implies \frac{1}{q(z,x)} \geq \frac{1}{q(z,y)} \frac{1}{q(y,x)} \implies q(x,z) \geq q(y,z)q(x,y)$ and therefore, we must have $q(x,z) = q(x,y)q(y,z)$.

Step 5. Finally, we argue that $\triangleright$ is asymmetric and transitive. As defined, $\triangleright$ is clearly asymmetric and compares any two distinct locations x and y such that $q(x,y)$ is rational. We show that it is also transitive. Suppose $x \neq y \neq z$ are such that $x \triangleright y$ and $y \triangleright z$. This implies that $q(x,y)$ and $q(y,z)$ are rational numbers. Let $q(x,y) = \frac{n_1}{\tilde{n}_1}$ and $q(y,z) = \frac{n_2}{\tilde{n}_2}$. Let R_M, R_L and R_K be peak-unanimous and non-overlapping profiles such that $p(R_M) = x$, $p(R_L) = y$ and $p(R_K) = z$, and $|M| = n_1 \times n_2$, $|L| = n_2 \times \tilde{n}_1$ and $|K| = \tilde{n}_1 \times \tilde{n}_2$. Consider the problem $(1, (R_M, R_L, R_K))$. We have $\frac{|M|}{|L|} = q(x,y)$ and $\frac{|L|}{|K|} = q(y,z)$. Since $x \triangleright y$, we have $R_M \succ R_L$ and since $y \triangleright z$ we have $R_L \succ R_K$. However, $\succ$ is almost transitive and therefore, it must be that $R_M \succ R_K$. This happens only if either $\frac{|M|}{|K|} > q(x,z)$ or $\frac{|M|}{|K|} = q(x,z)$ and $x \triangleright z$. But $\frac{|M|}{|K|} = \frac{n_1}{\tilde{n}_1} \times \frac{n_2}{\tilde{n}_2} = q(x,y)q(y,z) = q(x,z)$. So it must be that $x \triangleright z$. ■

5 Concluding Remarks

Richness: We conclude by illustrating the richness of the class of rules associated with weighted majoritarian priorities. A simple majoritarian priority (Example 3) is a weighted majoritarian priority if and only if it uses the same tie-breaking rule across all indifference classes and this tie-breaking rule is defined by a strict complete order $\triangleright$ on $\mathbb{R}$ such that for any n and any peak-unanimous and non-overlapping profiles $R_L, R_K \in T_n$, we have $R_L \succ_n R_K \iff p(R_L) \triangleright p(R_K)$.[11] The same is true for a two regions majoritarian priority (Example 6), i.e., it is a weighted majoritarian priority if and only if there exists a strict partial order $\triangleright$ on $\mathbb{R}$ such that for any v and any peak-unanimous and non-overlapping profiles $R_L, R_K \in T_v$, we have $R_L \succ_v R_K \iff p(R_L) \triangleright p(R_K)$.[12] Thus, in particular, the left majoritarian,

[11] The corresponding q is such that $q(x,y) = 1, \forall (x,y) \in \mathbb{R}^2$.

[12] The corresponding q is such that $q(x,y) = 1/\lambda_{x_0}$ if $x < x_0 \leq y$, $q(x,y) = \lambda_{x_0}$ if $y < x_0 \leq x$ and $q(x,y) = 1$ if either $x, y < x_0$ or $x, y \geq x_0$. Note that if λ_{x_0} is a rational number, the order $\triangleright$ is complete.

right majoritarian, left-two-regions majoritarian, and right-two-regions majoritarian priorities are weighted majoritarian priorities. Similarly, the centralist majoritarian priority (Example 7) with u such that there exists a decreasing positive function $\phi(d)$ such that $u(n,d) = n\phi(d)$, is a weighted majoritarian priority if the same tie-breaking rule is used across all indifference classes and is defined by a strict partial order on $\mathbb{R}$.[13]

Extensions: It is clear from Examples 5 and 8 that dropping either anonymity or peak-only from the characterization offered in Theorem 3 leads to a non-trivial enlargement of the class of rules. We offer a discussion on this issue in an online supplement where we also provide some partial characterizations of the classes of rules obtained when one drops either of the aforementioned axioms, or both.[14]

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[13] The corresponding q is such that $q(x,y) = \frac{\phi(|y-x_0|)}{\phi(|x-x_0|)}, \forall(x,y) \in \mathbb{R}^2$.

[14] The online supplement is available at http://staff.vwi.unibe.ch/bochet/papers/SupplementMajoritarian.pdf

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