

# A PROOF OF BLACKWELL'S THEOREM

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# A Proof of Blackwell's Theorem<sup>\*</sup>

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## Abstract

This note gives a new proof of Blackwell's celebrated result. The result is a bit stronger than the classical version since the action set and the prior are fixed, and only the utility of the decision maker varies. I show directly that a decision maker has access to a larger set of joint distributions over actions and states of the world if and only if her information improves in the garbling order.

## 1 Introduction

This note provides a proof of Blackwell's theorem (Blackwell, 1951, 1953). If a decision maker is identified with a prior on the states of the world, an action set, and a utility function over actions and states of the world, Blackwell's theorem says that an experiment  $\pi$ , that provides information about the state of the world, is preferred by every decision maker to an experiment  $\pi'$  if and only if  $\pi'$  is a garble of  $\pi$ . The proof I provide is relatively simple, and has the merit of making the intuitive point that the choice set of the decision maker is enlarged by moving from  $\pi'$  to  $\pi$  if and only if  $\pi'$  is a garble of  $\pi$ , which is absent from other proofs (Blackwell, 1951, 1953; Ponsard, 1975; Cremer, 1982; Leshno and Spector, 1992). Another advantage of this proof is that it varies only the utility of the decision maker, and not the prior or the action

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set<sup>1</sup>, so the difficult direction of the result ( $\pi$  more useful than  $\pi'$  implies that  $\pi'$  is a garble of  $\pi$ ) is slightly stronger here than in Blackwell's original formulation.

## 2 Setup and Preliminary Results

There is a finite action set  $A$ , with  $|A| \geq 2$ , and a finite set of states of the world  $\Omega$ . The prior is a probability distribution  $p(\omega)$  in  $\Delta(\Omega)$ . The payoff of the decision maker is given by a real valued payoff function  $u(a, \omega)$ . Let  $U$  be the set of such payoff functions. An experiment is given by a random variable  $x$ , with finite support  $X$  and a joint distribution function  $\pi$  on  $X \times \Omega$  with marginal  $p(\cdot)$  on  $\Omega$ . When the decision maker can observe the realization of  $x$ , but not that of  $\omega$ , she has access to mixed strategies  $\sigma(a|x)$ , with  $\sum_a \sigma(a|x) = 1$  for all  $x$ . Let  $\Sigma(\pi)$  be the set of strategies accessible to a decision maker endowed with experiment  $\pi$ . Ultimately, the decision maker only cares about the joint distributions of actions and states of the world,  $\varphi(a, \omega)$ . Let  $\Phi(\pi)$  be the set of joint distributions she can generate when endowed with  $\pi$ , or *policy space*. It is restricted by her lack of knowledge in the following way:

$$\Phi(\pi) = \left\{ \varphi(a, \omega) : \exists \sigma \in \Sigma(\pi), \varphi(a, \omega) = \sum_x \sigma(a|x) \pi(x, \omega) \right\}.$$

It is easy to show that this set is a compact, and convex subset of  $[0, 1]^{|A| \times |\Omega|}$ . Then the problem of decision maker  $u$  endowed with experiment  $\pi$  is given by the following linear program

$$V(\pi, u) = \max_{\varphi \in \Phi(\pi)} \sum_{a, \omega} \varphi(a, \omega) u(a, \omega).$$

**Definition 1** (Usefulness Order). *I say that an experiment  $\pi$  is more useful than another experiment  $\pi'$ , and write  $\pi \succeq \pi'$ , if all decision makers get a higher value when endowed with  $\pi$  than when they are endowed with  $\pi'$ , that is,*

$$\pi \succeq \pi' \Leftrightarrow V(\pi, u) \geq V(\pi', u), \forall u \in U$$

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<sup>1</sup>Leshno and Spector (1992) also fix the action set, but use a different proof technique based on matrices.

**Definition 2** (Garbling Order). *I say that  $\pi'$  is a garble of  $\pi$ , and write  $\pi' \trianglelefteq \pi$  if there exists a function  $f : X \times X' \rightarrow [0, 1]$  such that  $\pi'(x', \omega) = \sum_x f(x, x')\pi(x, \omega)$  and  $\sum_{x'} f(x, x') = 1$  for all  $x$ . Two experiments  $\pi$  and  $\pi'$  are equivalent, denoted by  $\pi \sim \pi'$ , if  $\pi \trianglelefteq \pi'$  and  $\pi' \trianglelefteq \pi$ .*

Note that this definition provides a different interpretation of  $\Phi(\pi)$  as the set of garbles  $\pi'$  of  $\pi$  such that  $|X'| \leq |A|$ . For each function  $f$  satisfying the conditions of the definition, I will denote by  $f \circ \pi$  the corresponding garble of  $\pi$ .

It is useful to prove a few basic results about garbles. The first of these results shows that, if one can observe two experiments  $f_1 \circ \pi$  with realization space  $X_1$ , and  $f_2 \circ \pi$  with realization space  $X_2$ , which are both garbles of  $\pi$ , then the experiment  $f_1 f_2 \circ \pi$ , with realization space  $X_1 \times X_2$ , is also a garble of  $\pi$ . This easily extends to a finite number of garbles.

**Lemma 1.** *Let  $f_1 \circ \pi, \dots, f_K \circ \pi$  be garbles of  $\pi$ . Then*

$$f_1 \cdots f_K \circ \pi \trianglelefteq \pi.$$

*Proof.* Let  $g(x, x_1, \dots, x_K) = f_1(x, x_1) \cdots f_K(x, x_K)$ . Then

$$\sum_{x_1, \dots, x_K} g(x, x_1, \dots, x_K) = \sum_{x_1} f_1(x, x_1) \cdots \sum_{x_K} f_K(x, x_K) = 1.$$

so that  $f_1 \cdots f_K \circ \pi$  is indeed a garble of  $\pi$ . □

The second result shows that if one is allowed to combine experiments from the set of binary garbles of  $\pi$ , i.e. garbles with support of size 2, then one can reconstitute all the information in  $\pi$ . The idea is simple: for each possible realization  $x$  of  $\pi$ , define the binary garble of  $\pi$  that returns 1 if  $x$  is realized and 0 otherwise; then combining all these garbles gives exactly the same information as  $\pi$ . Without loss of generality, I can fix the set  $B = \{0, 1\}$ , and denote the set of binary garbles of  $\pi$  by

$$\Gamma_b(\pi) = \left\{ \pi'(x', \omega) : \exists f : X \times B \rightarrow \mathbb{R}^+, \pi'(x', \omega) = \sum_x f(x, x')\pi(x, \omega) \text{ and } \sum_{x' \in B} f(x, x') = 1 \right\}.$$

**Lemma 2** (Reconstitution from Binary Garbles). *Consider an experiment  $\pi$  with support  $X$ . Then there exists  $|X|$  binary garbles  $f_1 \circ \pi, \dots, f_{|X|} \circ \pi \in \Gamma_b(\pi)$  such that*

$$f_1 \cdots f_{|X|} \circ \pi \sim \pi.$$

*Proof.* Let  $x_1, \dots, x_{|X|}$  be the elements of  $X$ . Then let  $f_k(x, 1) = \mathbb{1}_{x=x_k}$  and  $f_k(x, 0) = 1 - f_k(x, 1)$ . The  $|X|$  functions thus defined satisfy the conditions of [Definition 2](#), so they generate  $|X|$  binary garbles  $f_1 \circ \pi, \dots, f_{|X|} \circ \pi$ . It is easy to see that  $(f_k \circ \pi)(1, \omega) = \pi(x_k, \omega)$ , therefore observing the combined outcomes of all the experiments  $f_1 \circ \pi, \dots, f_{|X|} \circ \pi$  intuitively allows to reconstitute the  $\pi$ . To show this formally consider the experiment  $f_1 \cdots f_{|X|} \circ \pi$ . Its realization space is  $\{0, 1\}^{|X|}$ , but in fact the only vectors that occur with positive probability are the vectors with 0 on every dimension except one. Let  $e^k$  be the vector with a 1 on the  $k$ -th dimension and zeros elsewhere. Then for every  $k = 1, \dots, |X|$

$$(f_1 \cdots f_{|X|} \circ \pi)(e^k, \omega) = \pi(x_k, \omega),$$

which proves that  $f_1 \cdots f_{|X|} \circ \pi \sim \pi$ . In fact,  $f_1 \cdots f_{|X|} \circ \pi$  is exactly  $\pi$ , up to a recoding of the set  $X$ . □

### 3 Blackwell's Theorem

Blackwell's theorem says that the usefulness order and the garbling order are the same. I decompose the proof in two steps. First, I show by classical arguments that an experiment is more informative than another if and only if it generates a larger policy space in the set containment order. Second, I show that an experiment generates a larger policy space than another one if and only if the latter is a garbling of the former. The latter part is the novel one and it relies on the binary decomposition result. The idea for the difficult implication is to show that, if  $\pi$  leads to a larger policy space than  $\pi'$ , then the binary reconstitution of  $\pi'$ , which is informationally equivalent to  $\pi'$ , is a garble of  $\pi$ .

**Theorem 1** (Blackwell).  $\pi \succeq \pi' \Leftrightarrow \Phi(\pi) \supseteq \Phi(\pi') \Leftrightarrow \pi \triangleright \pi'$ .

*Proof.* I write one lemma for each step.

**Lemma 3.**  $\pi \succeq \pi' \Leftrightarrow \Phi(\pi) \supseteq \Phi(\pi')$

*Proof.*  $\Leftarrow$  is due to the fact that maximizing a function over a larger set always yields a higher value.  $\Rightarrow$  is due to a separation theorem. Indeed, suppose that there exists a policy  $\varphi(a, \omega)$  in  $\Phi(\pi') \setminus \Phi(\pi)$ . Then because  $\Phi(\pi)$  is a closed convex set, the hyperplane separation theorem implies the existence of a vector  $u \in U$  such that  $\sum_{a, \omega} u(a, \omega) \varphi(a, \omega) > V(\pi, u)$ .  $\square$

**Lemma 4.**  $\Phi(\pi) \supseteq \Phi(\pi') \Leftrightarrow \pi \triangleright \pi'$

*Proof.*  $\Leftarrow$  is the more natural sense. Suppose that  $\pi'$  is a garble of  $\pi$ , and let  $f(\cdot)$  be the associated garbling function. Let  $\varphi \in \Phi(\pi')$  be the policy generated by the associated strategy  $\sigma \in \Sigma(\pi')$ . Consider the strategy

$$\hat{\sigma}(a|x) \equiv \sum_{x'} f(x, x') \sigma(a|x').$$

It is an element of  $\Sigma(\pi)$  since

$$\sum_a \hat{\sigma}(a|x) = \sum_{x'} f(x, x') \sum_a \sigma(a|x') = 1.$$

And I can write

$$\begin{aligned} \varphi(a, \omega) &= \sum_{x'} \sigma(a|x') \pi'(x', \omega) \\ &= \sum_{x'} \sigma(a|x') \sum_x f(x, x') \pi(x, \omega) \\ &= \sum_x \sum_{x'} f(x, x') \sigma(a|x') \pi(x, \omega) \\ &= \sum_x \hat{\sigma}(a|x) \pi(x, \omega), \end{aligned}$$

which shows that  $\varphi \in \Phi(\pi)$ .

For  $\Rightarrow$ , suppose  $\Phi(\pi') \subseteq \Phi(\pi)$ . Then, since  $|A| \geq 2$ , I have  $\Gamma_b(\pi') \subseteq \Phi(\pi') \subseteq \Phi(\pi)$ . Then, by Lemma 2, I can pick  $|X'|$  binary garbles  $f_1 \circ \pi', \dots, f_{|X'|} \circ \pi'$  in  $\Phi(\pi')$  that reconstitute  $\pi'$ , so that  $f_1 \cdots f_{|X'|} \circ \pi' \sim \pi'$ .

Since  $f_k \circ \pi' \in \Phi(\pi)$ ,  $f_k \circ \pi'$  is a garble of  $\pi$ , so it is possible to find a function  $g_k : X \times B \rightarrow [0, 1]$  such that  $g_k(x, 0) + g_k(x, 1) = 1$ , and  $f_k \circ \pi' = g_k \circ \pi$ .

Consider the function  $g : X \times B \rightarrow \mathbb{R}^+$  defined by  $g(x, b) = \sum_k g_k(x, b)$ . I can write

$$\begin{aligned}
\boxed{\sum_x g(x, 1)\pi(x, \omega)} &= \sum_x \sum_k g_k(x, 1)\pi(x, \omega) = \sum_k \sum_x g_k(x, 1)\pi(x, \omega) \\
&= \sum_k (\pi \circ g_k)(1, \omega) = \sum_k (\pi' \circ f_k)(1, \omega) \\
&= \sum_k \pi'(x_k, \omega) = p(\omega) \\
&= \boxed{\sum_x \pi(x, \omega)}
\end{aligned}$$

This can be seen as a system of  $|\Omega|$  equations in  $|X|$  unknowns, the  $(g(x, 1))_{x \in X}$ . It has at least one solution which is  $g(x, 1) = 1$ , for all  $x$ . If  $|X| \leq |\Omega|$ , this is the unique solution, and therefore the functions  $g_k(\cdot)$  must be such that  $g(x, 1) = 1$ , for all  $x$ .

Suppose instead that  $|X| > |\Omega|$ . Then I show that the  $g_k(\cdot)$  functions can be chosen so that  $g(x, 1) = 1$  for all  $x$ . To see this note first that because  $g_k(x, 1) + g_k(x, 0) = 1$ , the problem of finding the  $g_k(\cdot)$  functions can be reduced to solving the system of  $|X'| \times |\Omega|$  equations in  $|X'| \times |X|$  unknowns given by  $\sum_x \in X g_k(x, 1)\pi(x, \omega) = \pi'(x, \omega)$ , for each  $k = 1, \dots, |X'|$ , and each  $\omega \in \Omega$ . Because  $|X| > |\Omega|$  the system has multiple solutions. Adding the  $|X|$  equations  $g(x, 1) = \sum_k g_k(x, 1) = 1$  to the system leaves the number of equation below the number of unknowns, so we can indeed choose the  $g_k(\cdot \cdot \cdot)$  functions so that  $g(x, 1) = 1$ , for all  $x$ .

Knowing this, I show that  $f_1 \cdots f_{|X'|} \circ \pi'$  is a garble of  $\pi$  as follows. For every  $e \in \{0, 1\}^{|X'|}$ ,

I have

$$\begin{aligned}
(\pi' \circ f_1 \cdots f_{|X'|}) (e, \omega) &= \sum_k \mathbb{1}_{e=e^k} (\pi' \circ f_k) (1, \omega) \\
&= \sum_k \mathbb{1}_{e=e^k} (\pi \circ g_k) (1, \omega) \\
&= \sum_k \mathbb{1}_{e=e^k} \sum_x g_k(x, 1) \pi(x, \omega) \\
&= \sum_x \underbrace{\sum_k \mathbb{1}_{e=e^k} g_k(x, 1)}_{\equiv h(x,e)} \pi(x, \omega)
\end{aligned}$$

Hence, to prove that  $f_1 \cdots f_{|X'|} \circ \pi'$  is a garble of  $\pi$ , I just need to show that  $\sum_e h(x, e) = 1$ . To see this note that  $h(x, e) = 0$  if  $e$  is not one of the  $e^k$  vectors, and  $h(x, e^k) = g_k(x, 1)$ . Therefore,

$$\sum_e h(x, e) = \sum_k g_k(x, 1) = g(x, 1) = 1.$$

□

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